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Adomian decomposition method for analytical solution of a continuous arithmetic Asian option pricing model

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Abstract

One of the main issues of concern in financial mathematics has been a viable method for obtaining analytical solutions of the Black-Scholes model associated with Arithmetic Asian Option (AAO). In this paper, a proposed semi-analytical technique: Adomian Decomposition Method (ADM) is applied for the first time, for analytical solution of a continuous arithmetic Asian option model. The ADM gives the solution in explicit form with few iterations. The computational work involved is less. However, high level of accuracy is not neglected. The obtained solution conforms with those of Rogers and Shi (J. of Applied Probability 32: 1995, 1077-1088), and Elshegmani and Ahmad (ScienceAsia, 39S: 2013, 67–69). Thus, the proposed method is highly recommended for analytical solution of other versions of Asian option pricing models such as the geometric form for puts and calls, even in their time-fractional forms.

Keywords: *adomian decomposition method, asian option, black-scholes model, option pricing*

1. Introduction

In financial settings, option contracts are great tools for hedging and speculative leveraging. Thus, Asian option has its payoff being determined specifically by the corresponding underlying asset for a prescribed period of time (not on the price of the underlying). This makes it a unique form of option contract over other forms. Asian options are classified as path dependent [1-8] when compared with the other forms of options in practice such as lookback, American, European among others. Asian options are basically of two forms namely: the Arithmetic Asian Option (AAO), and the Geometric Asian Option (GAO) which is distinguished for a closed form solution.

However, obtaining closed form solution of the Arithmetic Asian Option has been a difficult task in the theory of option pricing [9-10]. This issue of concern has attracted the attention of so many authors and researchers leading to the development of solution techniques to that effect [11-17]. Meanwhile, other analytical, numerical, approximate or semi-analytical techniques of interest related to this study are those of [18-29]. Despite all these approaches, there is vital need for a more effective and efficient methods of solution which may be numerical, analytical or semi-numerical.

In this work, the Adomian Decomposition Method (ADM) is proposed as a semi-analytical method (for the first time in literature), for obtaining analytical solution of a continuous arithmetic Asian model for option pricing. The remaining parts of the paper are organized as follows: a brief remark on pricing model for Asian option is presented in section 2. The proposed ADM as method of solution is presented in section number 3; application and illustrative examples are presented in section 4. Lastly, concluding remark is contained in section 5.

2. The Pricing Model for Asian Option

Let the stock price at time t, denoted as $S(t)$ assumed to follow a Geometric (Exponential) Brownian Motion (GBM) be governed by the stochastic differential equation (or dynamic) of the form:

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$$dS(t)=S(t)\left(rdt+\sigma dW(t)\right),\ t\in[o,\infty) \quad (1)$$

with the following defined parameters: σ a volatility (or percentage) coefficient, and a mean rate of return, r, where $W(t),\ 0\le t\le T$ is a standard Wiener process. Then the payoff function for an Asian option [29-31] having an Arithmetic Average Strike (AAS) is defined and denoted by:

$$Q(T)=\left(0,\ S(T)-\frac{1}{T}\int_0^T S(\varsigma)d\varsigma\right)^+. \quad (2)$$

The price of the option at $0\le t\le T$ regarded as a risk-neutral framework for option pricing formulation is denoted as:

$$Q(t)=\mathrm{E}\left(\exp\left(-r(T-t)\right)Q(T)|F_t\right) \quad (3)$$

where $\mathrm{E}(\cdot)$ represents mathematical operator in expectation form and F_t a filtration.

The payoff function, $Q(T)$ is path-dependent. So, the introduction of the following stochastic process [27]:

$$\left\{I(t)=\int_0^T S(\varsigma)d\varsigma,\ S(0)=S_0\right. \quad (4)$$

where the function $I(t)$ is regarded as the strike price running sum. Whence, the associated model for Asian call option price is:

$$\frac{\partial Q}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 Q}{\partial S^2}+rS\frac{\partial Q}{\partial S}+S\frac{\partial Q}{\partial I}-rQ=0. \quad (5)$$

Model (5) is solved by $Q=Q(S,I,t)$ when considering the continuous form of arithmetic average strike. At $\alpha=1$, (5) acts as a particular case of the notable time-fractional Black-Scholes option pricing model, besides the averaging term: $\left(S\frac{\partial Q}{\partial I}\right)$. To obtain the solution of this model, development and adoption of numerical, approximate or semi-analytical methods will be considered [21-23]. It is obvious that (5) is a 3-dimensional model, so it will give computational problem. Therefore, the following transformation variables are introduced to reduce it to ta lower dimensional form [2, 24]:

$$\begin{cases} Q(S,I,t)=Sh(t,\psi), \\ \psi S=k-\dfrac{I}{T}. \end{cases} \quad (6)$$

As a result, (5) becomes:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 h}{\partial \psi^2}-\left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial h}{\partial \psi}=0. \\ h(T,\psi)=\varphi(\psi). \end{cases} \tag{7}$$

Show in (7) is now in 2-dimensional form. Through the relation in (6), the solution shall be applied to obtain the price of the Asian option.

3. The Analysis of the Adomian Decomposition Method [20, 21, 23]

Consider a general nonlinear nonhomogeneous partial differential equation of the form:

$$H\eta(x,t)+R\eta(x,t)+N\eta(x,t)=\phi(x,t) \tag{8}$$

where H is a differential operator (of first order) in t, the differential operator, R is regarded as linear, the differential operator, N signifies the nonlinear term, while the source term is denoted as $\phi(x,t)$. Suppose $L(\cdot)=H(\cdot)$ is invertible such that $L_t^{-1}(\cdot)=\int_o^t(\cdot)ds$ exists, then (8) becomes:

$$L\eta(x,t)=\phi(x,t)-\left(R\eta(x,t)+N\eta(x,t)\right). \tag{9}$$

So, operating $L_t^{-1}(\cdot)$ on both sides of (9) gives:

$$\eta(x,t)=\vartheta(x)+L_t^{-1}\{\phi(x,t)\}-L_t^{-1}\{\left(R\eta(x,t)+N\eta(x,t)\right)\} \tag{10}$$

where $\vartheta(x)$ represents the term arising from the application of the initial conditions depicted as:

$$\begin{cases} \vartheta(x)=\sum_{i=0}^{n} p_i t^i, \\ p_i \Rightarrow \text{initial conditions.} \end{cases} \tag{11}$$

By decomposition method, the solution, $\eta(x,t)$ is expressed as a sum of infinite series given as:

$$\eta(x,t)=\sum_{n=0}^{\infty}\eta_n(x,t) \tag{12}$$

where the nonlinear term is defined as:

$$N\left(\eta(x,t)\right)=\sum_{m=0}^{\infty}A_m. \tag{13}$$

In (13) the Adomian polynomials, A_m are defined as follows:

$$A_m = \frac{1}{m!}\frac{\partial^m}{\partial \upsilon^m}\left[N\left(\sum_{i=0}^{m} \upsilon^i \eta_i \right) \right]_{\upsilon=0}, \tag{14}$$

thus, using (12) and (13) in (10) gives:

$$\sum_{m=0}^{\infty} \eta_m(x,t) = \vartheta(x) + L_t^{-1}\{h(x,t)\} - L_t^{-1}\left\{ \left(N\left(\sum_{m=0}^{\infty} A_m \right) + R\left(\sum_{m=0}^{\infty} \eta_m(x,t) \right) \right) \right\}. \tag{15}$$

Hence, $\eta(t,x)$ as the required solution is obtained through the recursive system (relation):

$$\begin{cases} \eta_0 = \vartheta(x) + L_t^{-1}\{h(x,t)\} \\ \eta_{n+1} = -L_t^{-1}\{(R(\eta_n) + N(A_n))\}, n \geq 0 \end{cases} \tag{16}$$

and $\eta(t,x)$ is finally given as follows:

$$\eta(x,t) = \lim_{m \to \infty} \left(\sum_{m=0}^{\infty} \eta_m \right). \tag{17}$$

4. Applications and Illustrative examples

Here, the analytical solution(s) are considered based on the proposed ADM. Consider (5) via (6, 7) in the following form:

$$\begin{cases} \dfrac{\partial h}{\partial t} = \left(\dfrac{1}{T} + r\psi \right) \dfrac{\partial h}{\partial \psi} - \dfrac{1}{2}\sigma^2 \psi^2 \dfrac{\partial^2 h}{\partial \psi^2} \\ h(0,\psi) = \varphi(\psi). \end{cases} \tag{18}$$

In an operator form, (18) becomes:

$$\begin{cases} L_t h = \left(\dfrac{1}{T} + r\psi \right) L_\psi h - \dfrac{1}{2}\sigma^2 \psi^2 L_{\psi\psi} h \\ h(0,\psi) = \varphi(\psi). \end{cases}, \tag{19}$$

So, operating $L_t^{-1}(\cdot)$ on both sides of (19) gives:

$$L_t^{-1} h = \left(\frac{1}{T} + r\psi \right) L_t^{-1}(L_\psi h) - \frac{1}{2}\sigma^2 \psi^2 L_t^{-1}(L_{\psi\psi} h). \tag{20}$$

$$\Rightarrow h(\psi,t) = \left(\frac{1}{T} + r\psi \right) L_t^{-1}(L_\psi h) - \frac{1}{2}\sigma^2 \psi^2 L_t^{-1}(L_{\psi\psi} h). \tag{21}$$

By decomposing $h(\psi,t)$, we have:

$$\sum_{n=0}^{\infty} h_n(x,t) = h(x,0) + \left(\frac{1}{T} + r\psi \right) L_t^{-1}\left(L_\psi \left(\sum_{n=0}^{\infty} h_n(x,t) \right) \right) - \frac{1}{2}\sigma^2 \psi^2 L_t^{-1}\left(L_{\psi\psi} \left(\sum_{n=0}^{\infty} h_n(x,t) \right) \right). \tag{22}$$

Thus,

$$\begin{cases} g_0 = g(\psi,0), \\ g_{n+1} = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi g_n\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}g_n\right), n\geq 0. \end{cases} \tag{23}$$

Therefore, the recursive relation in (23) yields:

$$\begin{cases} h_0 = h(\psi,0) \\ h_1 = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi h_0\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}h_0\right) \\ h_2 = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi h_1\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}h_1\right) \\ h_3 = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi h_2\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}h_2\right) \\ h_4 = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi h_3\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}h_3\right) \\ h_5 = \left(\frac{1}{T}+r\psi\right)L_t^{-1}\left(L_\psi h_4\right)-\frac{1}{2}\sigma^2\psi^2L_t^{-1}\left(L_{\psi\psi}h_4\right) \\ \vdots . \end{cases} \tag{24}$$

The following are obtained by subjecting (18) to:

$$h(\psi,0)=\frac{1}{rT}\left(1-e^{-rT}\right)-\psi e^{-rT}. \tag{25}$$

$$\left.\begin{array}{l} h_1 = -\left(\frac{1}{T}+r\psi\right)te^{-rT}, \\ h_3 = -\frac{1}{3!}\left(\frac{1}{T}+r\psi\right)r^2t^3e^{-rT}, \\ h_5 = -\frac{1}{5!}\left(\frac{1}{T}+r\psi\right)r^4t^5e^{-rT}, \\ h_7 = -\frac{1}{7!}\left(\frac{1}{T}+r\psi\right)r^6t^7e^{-rT}, \\ h_9 = -\frac{1}{9!}\left(\frac{1}{T}+r\psi\right)r^8t^9e^{-rT}, \\ \vdots \end{array}\right\} \left\{\begin{array}{l} h_2 = -\frac{1}{2!}\left(\frac{1}{T}+r\psi\right)rt^2e^{-rT}, \\ h_4 = -\frac{1}{4!}\left(\frac{1}{T}+r\psi\right)r^3t^4e^{-rT}, \\ h_6 = -\frac{1}{6!}\left(\frac{1}{T}+r\psi\right)r^5t^6e^{-rT}, \\ h_8 = -\frac{1}{8!}\left(\frac{1}{T}+r\psi\right)r^7t^8e^{-rT}, \\ h_{10} = -\frac{1}{10!}\left(\frac{1}{T}+r\psi\right)r^9t^{10}e^{-rT}, \\ \vdots \end{array}\right. \tag{26}$$

with a general recursive relation:

$$h_k = -\frac{1}{k!}\left(\frac{1}{T}+r\psi\right)r^{k-1}t^ke^{-rT},\ k\geq 1. \tag{27}$$

Hence,

$$
\begin{aligned}
h(\psi,t) &= \sum_{p=0}^{\infty} h_p \\
&= \left(-\psi e^{-rT} + \frac{1}{rT}\left(-e^{-rT}+1\right)\right) - \left(t + \frac{t^2 r}{2!} + \frac{t^3 r^2}{3!} + \frac{t^4 r^3}{4!} + \cdots\right)\left(\frac{1}{T} + r\psi\right)e^{-rT} \\
&= \begin{pmatrix} \left((rT)^{-1}\left(1-\exp(-rT)\right) - \psi \exp(-rT)\right) \\ -\frac{1}{r}\left(rt + \frac{(rt)^2}{2!} + \frac{(rt)^3}{3!} + \frac{(rt)^4}{4!} + \cdots\right)\left(\frac{1}{T} + r\psi\right)e^{-rT} \end{pmatrix} \\
&= \left(\frac{1}{rT}\left(1-\exp(-rT)\right) - \psi e^{-rT}\right) - \frac{1}{r}\sum_{i=1}^{\infty}\frac{(rt)^i}{i!}\left(\frac{1}{T} + r\psi\right)e^{-rT} \\
&= \left(\frac{1}{rT}\left(1-\exp(-rT)\right) - \psi e^{-rT}\right) - \frac{1}{r}\left(-1 + \sum_{i=0}^{\infty}\frac{(rt)^i}{i!}\right)\left(\frac{1}{T} + r\psi\right)e^{-rT} \\
&= \left(\frac{1}{rT}\left(1-\exp(-rT)\right) - \psi e^{-rT}\right) - \frac{1}{r}\left(-1 + e^{rt}\right)\left(\frac{1}{T} + r\psi\right)e^{-rT} \\
&= \left(\frac{1}{rT}\left(1-\exp\left(-r(T-t)\right)\right) - \psi e^{-r(T-t)}\right),\ \psi \le 0.
\end{aligned}
\tag{28}
$$

Meanwhile, from (6) we recall that:

$$
\begin{cases} Q(S,I,t) = Sh(t,\psi), \\ \psi ST = kT - I. \end{cases}
$$

$$
\therefore\ Q(S,I,t) = \left(\frac{S}{rT}\left(-\exp\left(-r(T-t)\right)+1\right) - \left(k - \frac{I}{T}\right)\exp\left(-r(T-t)\right)\right). \tag{29}
$$

Show in (29) gives the required solution of (5) (in analytical form) corresponding to the continuous arithmetic Asian option (CAAO) pricing model.

5. Conclusion

In this paper, we have successfully applied a proposed semi-analytical method: Adomian Decomposition Method (ADM) for obtaining analytical solution of the continuous arithmetic Asian option model. The application of ADM to the CAAO pricing model is done for the first time. The method involved less computational work without compromising the level of accuracy. The works of Rogers & Shi [1], and Elshegmani & Ahmad [2] serve as yardsticks to this current work.The proposed semi-analytical method is highly recommended for analytical solution of other versions of Asian option pricing models such as the geometric form for puts and calls. In addition, other financial PDEs resulting from stochastic dynamics can be considered via this approach. Future related work can involve the use of coupled ADM, restarted ADM, and the Laplace-Sumudu ADM (LSADM) for accuracy and speed comparison. Conflict of Interests: The authors declare no existence of conflict of interest regarding the publication of this paper.

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